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2004 J. Phys. A: Math. Gen. 37 L113

(http://iopscience.iop.org/0305-4470/37/10/L02)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) L113–L115 PII: S0305-4470(04)75362-7

LETTER TO THE EDITOR

Another integrable case in the Lorenz model

Tat-Leung Yee[1] **and Robert Conte**

Service de physique de l'état condensé (URA 2464), CEA-Saclay,
F-91191 Gif-sur-Yvette Cedex, France

E-mail: TonYee@ust.hk and Conte@drecam.saclay.cea.fr

Received 29 January 2004
Published 24 February 2004
Online at stacks.iop.org/JPhysA/37/L113 (DOI: 10.1088/0305-4470/37/10/L02)

Abstract
A scaling invariance in the Lorenz model allows one to consider the usually discarded case $\sigma = 0$. We integrate it with the third Painlevé function.

PACS numbers: 02.30.−f, 05.45.+b, 47.27.−i

1. Introduction

The Lorenz model [1]

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x) \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = rx - y - xz \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = xy - bz \tag{1}$$

in which (b, σ, r) are real constants, is a prototype of chaotic behaviour [3]. In particular, it fails the Painlevé test unless the parameters obey the constraints [4]

$$Q_2 \equiv (b - 2\sigma)(b + 3\sigma - 1) = 0 \tag{2}$$

$$\begin{aligned}\forall x_2 : Q_4 \equiv & -4\mathrm{i}(b - \sigma - 1)(b - 6\sigma + 2)x_2 - \tfrac{4}{3}(b - 3\sigma + 5)b\sigma r \\ & + (-4 + 10b + 30b^2 - 20b^3 - 16b^4)/27 \\ & + \left(-38b - 56b^2 - \tfrac{28}{3}b^3 + 88\sigma + 86b^2\sigma\right)\sigma/3 \\ & - 32\sigma/9 + 70b\sigma^2 - 64\sigma^3 - 58b\sigma^3 + 36\sigma^4 = 0.\end{aligned} \tag{3}$$

This system (2)–(3) depends on r only through the product $b\sigma r$, as a consequence of an obvious scaling invariance in the model, and it admits four solutions,

$$(b, \sigma, b\sigma r) = (1, 1/2, 0), (2, 1, 2/9), (1, 1/3, 0), (1, 0, 0). \tag{4}$$

In the first three cases, i.e. when the system (1) is nonlinear, which excludes $\sigma = 0$, the system can be explicitly integrated [4], and the general solution (x, y, z) is a single-valued function

[1] Permanent address: Department of Mathematics, The Hong-Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong.

of time expressed with, respectively, an elliptic function, the second and the third Painlevé functions.

In this letter, we consider the fourth case

$$(b, \sigma, r) = (1, 0, r). \tag{5}$$

The apparently linear nature of the dynamical system can be removed by eliminating y and z and considering the third-order differential equation for $x(t)$ [5],

$$y = x + x'/\sigma \qquad z = r - 1 - [(\sigma + 1)x' + x'']/(\sigma x) \tag{6}$$

$$xx''' - x'x'' + x^3x' + \sigma x^4 + (b + \sigma + 1)xx'' + (\sigma + 1)(bxx' - x'^2) + b(1 - r)\sigma x^2 = 0 \tag{7}$$

which also depends on r only through the product $b\sigma r$, and thus implements the above-mentioned scaling invariance. The necessary conditions for (7) to pass the Painlevé test are the same ($Q_2 = 0$, $Q_4 = 0$) as for the dynamical system (1), the restriction $\sigma \neq 0$ being now removed.

2. Integration for $b = 1, \sigma = 0$

Because of the scaling invariance, the following first integral [4] of the dynamical system (1),

$$(b, \sigma, r) = (1, \sigma, 0): \qquad K_3 = (y^2 + z^2)\,\mathrm{e}^{2t} \tag{8}$$

is also a first integral of the third-order equation for $(b, \sigma, b\sigma r) = (1, \sigma, 0)$, which includes the particular case of interest to us $(b, \sigma, b\sigma r) = (1, 0, 0)$,

$$(b, \sigma, b\sigma r) = (1, 0, 0): \qquad K^2 = \lim_{\sigma \to 0} \sigma^2 K_3 = \left[\left(\frac{x'' + x'}{x}\right)^2 + x'^2\right]\mathrm{e}^{2t}. \tag{9}$$

For $K = 0$, the general solution is

$$x = \mathrm{i}k \tanh \frac{k}{2}(t - t_0) - \mathrm{i} \qquad \mathrm{i}^2 = -1 \qquad (k, t_0)\ \text{arbitrary}. \tag{10}$$

For $K \neq 0$, after taking the usual parametric representation

$$\frac{x'' + x'}{x} = K\,\mathrm{e}^{-t}\cos\lambda \qquad x' = K\,\mathrm{e}^{-t}\sin\lambda \tag{11}$$

the second-order ODE for $\lambda(t)$ is found to be

$$\lambda'' - K\,\mathrm{e}^{-t}\sin\lambda = 0 \tag{12}$$

with the link

$$x(t) = \lambda'(t). \tag{13}$$

In the variable $\mathrm{e}^{\mathrm{i}\lambda}$, the differential equation (12) becomes algebraic and belongs to an already integrated class [2]. The overall result is the general solution

$$x = \mathrm{i} + 2\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\log w(\xi(t)) \qquad \mathrm{i}^2 = -1 \qquad \xi = a\,\mathrm{e}^{-t} \tag{14}$$

in which $w(\xi)$ is the particular third Painlevé function defined by

$$\frac{\mathrm{d}^2 w}{\mathrm{d}\xi^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}\xi}\right)^2 - \frac{\mathrm{d}w}{\xi\,\mathrm{d}\xi} + \frac{\alpha w^2 + \gamma w^3}{4\xi^2} + \frac{\beta}{4\xi} + \frac{\delta}{4w} \tag{15}$$

$$\alpha = 0 \qquad \beta = 0 \qquad \gamma\delta = -(K/a)^2. \tag{16}$$

3. Conclusion

Out of the two cases selected by the condition $Q_2 = 0$, one admits a first integral [4],

$$b = 2\sigma: \qquad K_1 = (x^2 - 2\sigma z)\,e^{2\sigma t} \tag{17}$$

but, in the second case $b = 1 - 3\sigma$, the first integral whose existence has been conjectured [6] is not yet known. The present result, which belongs to this unsettled case $b = 1 - 3\sigma$, should help to solve this open question.

Acknowledgments

T-LY thanks the Croucher Foundation for a postdoc grant at CEA. RC acknowledges support of the France–Hong Kong PROCORE grant 04807SF, which allowed this work to be started.

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